Coherent States and the Bargmann Representation in Relativistic Quantum Mechanics

Rishindra Melanathuru[1], D. C. Brody[2], L. P. Hughston[3] and G. W. Gibbons[4]

[1]vrm18@ic.ac.uk, [2]D.Brody@surrey.ac.uk, [3]L.Hughston@gold.ac.uk, [4]G.W.Gibbons@damtp.cam.ac.uk

Introduction

The phase space of a relativistic system can be identified with the future tube of complexified Minkowski space. A Hilbert space of square-integrable holomorphic functions can be constructed on the relativistic phase space, upon which a quantum measurement theory of spacetime phase-space events can be formulated. This Hilbert space of holomorphic functions can alternatively be interpreted as the Husimi (Fushimi) representation of relativistic quantum theory. A transformation analogous to that of Bargmann can then be worked out, leading to a new formulation of relativistic quantum theory in which the coherent states associated with the conformal group play an important role.

Phase space in relativistic mechanics

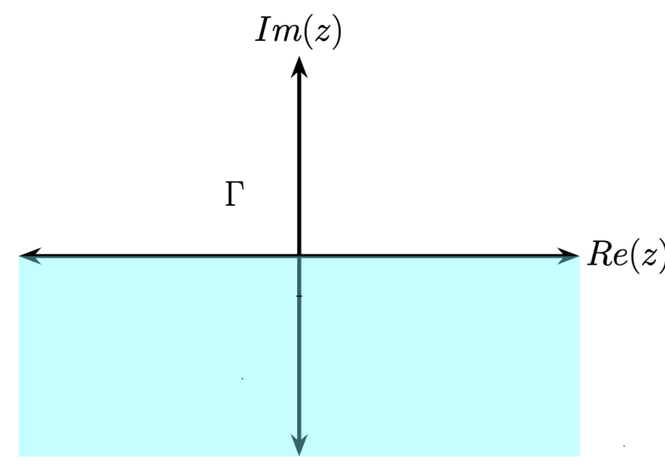

Figure 1: Future tube in one timelike dimension.

The phase space of a relativistic system is identified with the future tube Γ of complexified Minkowski space $\mathbb{CM}$, which is a submanifold of $\mathbb{CM}$ described by points of the form $z^a = x^a - ir^a$ where r^a is timelike and future pointing. The Kelvin transform [**1**] relates r^a to the momentum p^a by

$$r^a = \frac{\hbar\, p^a}{p_c p^c}, \qquad p^a = \frac{\hbar\, r^a}{r^c r_c}. \tag{1}$$

This induces a positive definite metric and its inverse

$$g_{ab} = -\frac{1}{r_c r^c}\left(\eta_{ab} - \frac{2\, r_a r_b}{r_c r^c}\right), \qquad k^{ab} = -r_c r^c\left(\eta^{ab} - \frac{2\, r^a r^b}{r_c r^c}\right) \tag{2}$$

on Γ, where $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$, and Hamilton's equations are

$$\frac{dx^a}{ds} = -g^{ab}\frac{\partial H}{\partial r^b}, \qquad \frac{dr^a}{ds} = k^{ab}\frac{\partial H}{\partial x^b}. \tag{3}$$

Relativistic quantum mechanics

The space $\mathcal{H} = \mathcal{L}^2(\Gamma, \mathcal{O})$ of square integrable holomorphic functions on the future tube can be viewed as the Hilbert space of quantum states on the phase space [**1**]. A general quantum state with density matrix $\rho(z,\bar{w})$ is defined to have the following properties:

1. $\rho(z,\bar{w})$ is Hermitian, i.e. $\rho(z,\bar{w}) = \bar{\rho}(\bar{w}, z)$.
2. $\rho(z,\bar{w})$ is positive, that is

$$\iint \bar{\Psi}(\bar{z})\,\rho(z,\bar{w})\Psi(w)\; d\mu_z\, d\mu_w \geq 0, \tag{4}$$

for any $\Psi(z) \in \mathcal{H}$, where $d\mu_z$ is Leb(Γ). The integrals each range over Γ.

3. $\rho(z,\bar{w})$ has unit trace, that is

$$\iint K(z,\bar{w})\,\rho(w,\bar{z})\; d\mu_w\, d\mu_z = \int \rho(z,\bar{z})\; d\mu_z = 1, \tag{5}$$

where $K(z,\bar{w})$ is the reproducing kernel (Bergman kernel) on $\mathcal{H}$ [**2**]. A pure state is a density matrix of the form $\rho(z,\bar{w}) = \Psi(z)\bar{\Psi}(\bar{w})$.

Quantum measurement

To make sense of quantum detection we define a positive operator valued measure (POVM) on Γ by setting

$$\phi_A(z,\bar{w}) = \int_{u\in A} K(z,\bar{u})K(u,\bar{w})\; d\mu_u, \tag{6}$$

where A is any element of the Borel σ-algebra on Γ. The probability that an event lies in this set is given by

$$\mathrm{Prob}(A) = \iint_{\Gamma} \phi_A(z,\bar{w})\,\rho(w,\bar{z})\; d\mu_w\, d\mu_z = \int_{z\in A} \rho(z,\bar{z})\; d\mu_z. \tag{7}$$

If the measurement outcome is the phase-space point w, then the output state is the coherent state $\Psi_w(z) = [K(w,\bar{w})]^{-1/2}K(z,\bar{w})$.

Phase-space localization

On phase space, no wave function can be sharply peaked. In particular, the maximum value that a squared wave function can take is given by

$$\max_{z\in\Gamma} \rho(z,\bar{z}) = \frac{3}{4\pi^4 \lambda\!\!\!^{-}{}_z^8}, \tag{8}$$

where $\lambda\!\!\!^{-}{}_z$ is the reduced Compton wavelength associated to the phase-space point z at which the maximum occurs.

Conformal transformations

The action of a Poincaré transformation on $z \in \Gamma$ is given by

$$z^a \to \Lambda^a_b z^b + B^a, \tag{9}$$

where Λ^a_b gives a Lorentz transform and B^a denotes a real four vector. The unitary representation of the Poincaré group acting on $\mathcal{H}$ is given by

$$U(z,\bar{w}) = K(\Lambda^a_b z^b + B^a, \bar{w}^a), \tag{10}$$

where K is the Bergman kernel. More generally, the Bergman kernel can be used to describe the full 15-parameter conformal group over $\mathcal{H}$ of which the Poincaré group is a proper subgroup [**1**].

Coherent states

The coherent state centred at $w \in \Gamma$, more specifically, is

$$\Psi_w(z) = \frac{8\sqrt{3}}{\pi^2}\frac{[\eta_{ab}(w^a-\bar{w}^a)(w^b-\bar{w}^b)]^2}{[\eta_{ab}(z^a-\bar{w}^a)(z^b-\bar{w}^b)]^4}. \tag{11}$$

The action of the conformal group on $\mathcal{H}$ leaves the space of coherent states invariant.

Husimi representation

We can identify $\mathcal{H}$ as a Husimi representation of relativistic quantum mechanics; that is to say, the elements of $\mathcal{H}$ are the Bargmann transforms [**3**] of the elements of $\mathcal{L}^2(V^+)$, where V^+ is a forward light cone. We construct a Bargmann transform between $\mathcal{H}$ and $\mathcal{L}^2(V^+)$. Using the inverse Bargmann transform $A(q,\bar{z})$ we can show that the associated coherent state $\psi_w(q) \in \mathcal{L}^2(V^+)$ is given by

$$\psi_w(q) = \int A(q,\bar{z})\,\Psi_w(z)\; d\mu_z = \frac{1}{\pi\sqrt{2^6 4!}}\left(\frac{(w-\bar{w})^2}{w^2\,\bar{w}^2}\right)^2 \exp\left(-\frac{\mathrm{i} q_a \bar{w}^a}{\bar{w}_b \bar{w}^b}\right). \tag{12}$$

Thus, a coherent state associated to the conformal group on V^+ is a plane wave that is exponentially damped into the future orientation.

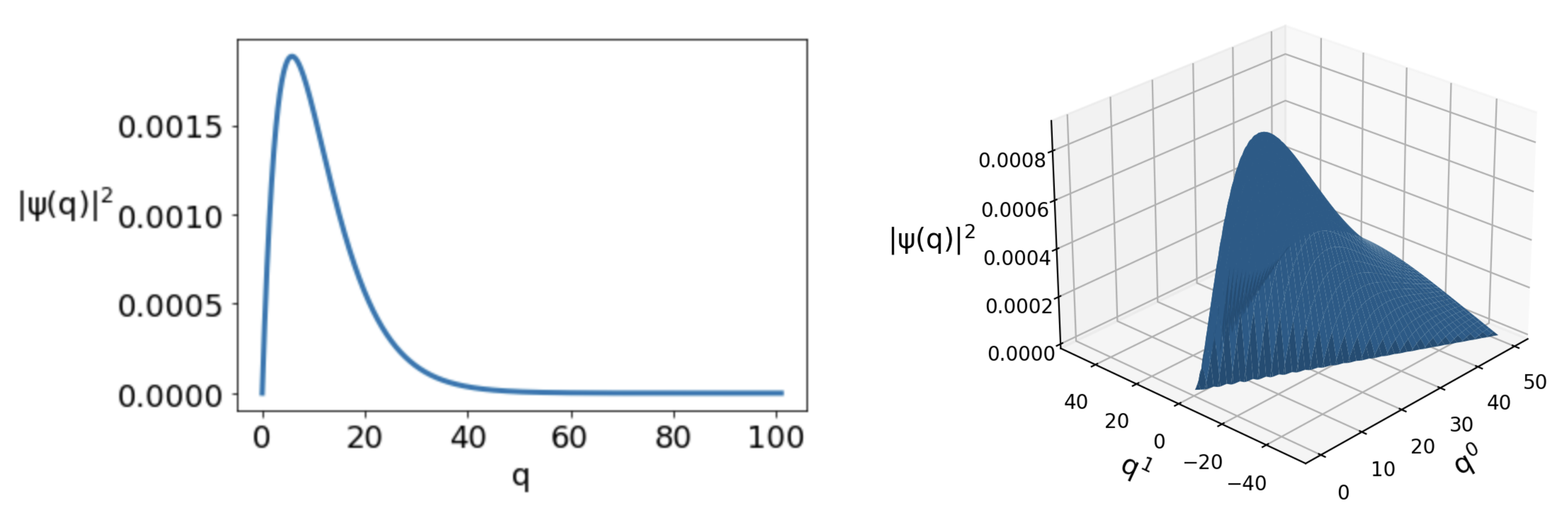

Figure 2: Probability densities of coherent states in 0 + 1 and 1 + 1 dimensions.

References

[1] D. C. Brody & L. P. Hughston (2021) *Quantum measurement of space-time events*, J. Phys. A: Math. Theor. **54**, 235304.
[2] S. Bergman (1970) *The Kernel Function and Conformal Mapping*. Providence, Rhode Island: American Mathematical Society.
[3] V. Bargmann (1961) *On a Hilbert space of analytic functions and an associated integral transform*. Commun. Pure and Applied Math **14**, 187-214.